Reinvent λ calculus by yourself

基础软件理论与实践公开课

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λ calculus, formally

Assume given an infinite set $\mathcal{V}$ of variables, denoted by x, y, z, etc. The set of lambda terms Λ is given by

$$\text{Lambda terms:} \qquad M, N ::= x \mid (MN) \mid (\lambda x.\, M)$$

The following are some examples of lambda terms:

$$(\lambda x.\, x) \qquad ((\lambda x.\, (xx))(\lambda y.\, (yy))) \qquad (\lambda f.\, (\lambda x.\, (f(fx))))$$

```
type rec t =
    | Var (string)
    | App (t,t)
    | Fun (string, t)
```

Where is the compuation happening ?

- β-reduction (function application in an informal sense)
- **β-redex** is a term of the form $(\lambda x.\, M)N$
- A β-redex **reduces** to $M[N/x]$

For example,

$$
\begin{aligned}
(\lambda x.\, y)(\underline{(\lambda z.\, zz)(\lambda w.\, w)}) \quad &\to_\beta \quad (\lambda x.\, y)(\underline{(\lambda w.\, w)(\lambda w.\, w)}) \\
&\to_\beta \quad \underline{(\lambda x.\, y)(\lambda w.\, w)} \\
&\to_\beta \quad y.
\end{aligned}
$$

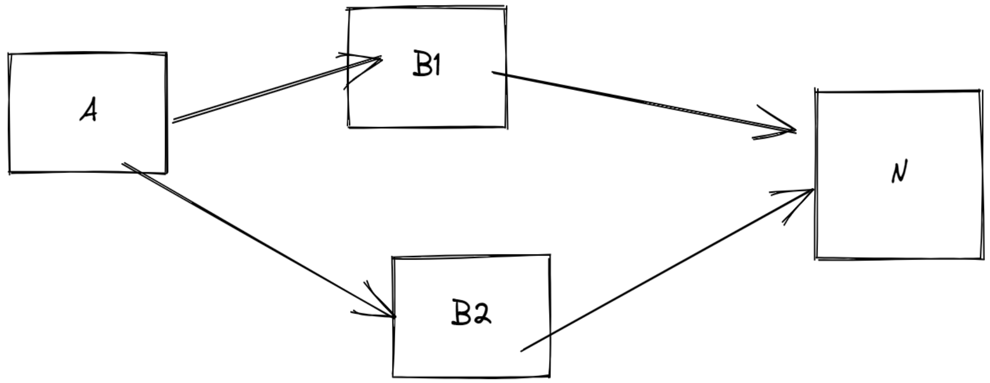
The same term can be reduced differently,

$$
\underline{(\lambda x.\, y)((\lambda z.\, zz)(\lambda w.\, w))} \quad \to_\beta \quad y
$$

Confluence of untyped lambda calculus

- Church-Rosser theorem

If there are reduction sequences from any term A to two different terms B_1 and B_2, then there exist reduction sequences from those two terms to some common term N.



- Result of computation is independent of the evaluation order
- Lanuages can choose different evaluation order, e.g, Haskell, ReScript

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Interpreter (natural semantics)

- Evaluate closed term, call by value

```
let rec eval = (t: lambda) => {
  switch t {
  | Var(_) => assert false
  | Fn(_, _) => t
  | App(f, arg) => {
    let Fn(x, body) = eval(f)
    let va = eval(arg)
    eval(subst(x, va, body)) // substitution explained later
    }
  }
}
```

A formal view of the natural semantics

- Call by value:

$$\frac{}{(\lambda x.\, a)\ \color{red}{v} \rightarrow a[v/x]}$$

- Left to right

$$\frac{a \rightarrow a'}{a\ b \rightarrow a'\ b} \qquad \frac{b \rightarrow b'}{\color{red}{v}\ b \rightarrow v\ b'}$$

- What are values?
  - functions
  - can we use functions to represent constant numbers, boolean value, etc?

Two Interpreters

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Eval using substitution

```
let rec eval = (t: lambda) => {
  switch t {
  | Var(_) => assert false
  | Fn(_, _) => t
  | App(f, arg) => {
    let Fn(x, body) = eval(f)
    let va = eval(arg)
    eval(subst(x, va, body))
    }
  }
}
```

Eval using env map

```
let eval = (t: lambda) => {
  let rec go = (e, t) => {
    switch t {
    | Var(x) => List.assoc(x, e)
    | Fn(x, body) => Vclosure(e, x, body)
    | App(f, arg) => {
      let Vclosure(e', x, body) = go(e, f)
      let va = go(e, arg)
      go(list{(x, va), ...e'}, body)
      }
    }
  }
  go (list{}, t)
}
```

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Two kinds of interpreters

- Substitution: **eagerly** replace the bound variables with the argument
- Environment: save the argument and **lazily** replace the bound variables
- Evaluate to the equivalent results

For lambda terms M without free variables

$$\textsf{eval1}(M) = \textsf{Fn}(x, N) \Leftrightarrow \textsf{eval2}(M) = \textsf{Vclosure}([], x, N)$$

Primitives

- Definition of boolean

$$\text{if\_then\_else}\ \bar{T}\ M\ N \twoheadrightarrow_\beta M$$
$$\text{if\_then\_else}\ \bar{F}\ M\ N \twoheadrightarrow_\beta N$$

- Boolean values

$$\begin{aligned} \bar{T} &= \lambda xy.\,x \\ \bar{F} &= \lambda xy.\,y \end{aligned}$$

- If-Then-Else

$$\text{if\_then\_else} = \lambda x.\,x$$

Church numerals

- The Church numerals $\bar{0}$, $\bar{1}$, $\bar{2}$, ... are defined by

$$\bar{n} \quad = \quad \lambda f x.\, f^n(x).$$

- Here are the first few Church numerals:

$$\begin{array}{rcl}
\bar{0} & = & \lambda f x.\, x \\
\bar{1} & = & \lambda f x.\, f x \\
\bar{2} & = & \lambda f x.\, f(f x) \\
\bar{3} & = & \lambda f x.\, f(f(f x)) \\
& \cdots &
\end{array}$$

Constructors

- Iso-morphism, used in Coq for number theory, called Peano number

```
type rec nat = Z | S(nat)
let three = S (S (S Z))
```

- Compare with the church numeral: $\bar{3} = \lambda fx.\ f(f(fx))$
- There is correspondence between constructors S and Z and bound variables f and x
- Recommended reading: Church encoding and Scott encoding

Arithmetic functions

$$\begin{aligned}\text{succ} &= \lambda nfx.\, f(nfx) \\ \text{add} &= \lambda nmfx.\, nf(mfx)\end{aligned}$$

Example

$$\begin{aligned}\text{succ}\ \bar{n} &= (\lambda nfx.\, f(nfx))(\lambda fx.\, f^n x) \\ &\to_\beta \lambda fx.\, f((\lambda fx.\, f^n x) fx) \\ &\twoheadrightarrow_\beta \lambda fx.\, f(f^n x) \\ &= \lambda fx.\, f^{n+1} x \\ &= \overline{n+1}\end{aligned}$$

More primitives

- Test whether a number is zero

$$\text{iszero} = \lambda n.\, n\ (\lambda z.\, \bar{F})\ \bar{T}$$

- Pair

$$\begin{aligned} \text{pair} &= \lambda xyz.\, z\ x\ y \\ \text{fst} &= \lambda p.\, p\ (\lambda xy.\, x) \\ \text{snd} &= \lambda p.\, p\ (\lambda xy.\, y) \end{aligned}$$

- Predecessor (simple version using pair)

$$\text{pred} = \lambda n.\, \text{fst}\ (n\ (\lambda p.\, \text{pair}(\text{snd}\ p)(\text{succ}(\text{snd}\ p)))(\text{pair}\ \bar{0}\ \bar{0}))$$

Pred for church numerals

$$
\begin{aligned}
\mathrm{f} &= \lambda p.\ \ \mathrm{pair}\ \ (\mathrm{second}\ \ p)\ (\mathrm{succ}\ \ (\mathrm{second}\ \ p)) \\
\mathrm{zero} &= (\lambda f. \lambda x.\ \ x) \\
\mathrm{pc0} &= \mathrm{pair}\ \ \mathrm{zero}\ \ \mathrm{zero} \\
\mathrm{pred} &= \lambda n.\ \ \mathrm{first}\ \ (n\ \ \mathrm{f}\ \ \mathrm{pc0})
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{pred\,three} &= \ \mathrm{first}\ \ (\mathrm{f}\ \ (\mathrm{f}\ \ (\mathrm{f}\ \ (\mathrm{pair}\ \ \mathrm{zero}\ \ \mathrm{zero})))) \\
&= \ \mathrm{first}\ \ (\mathrm{f}\ \ (\mathrm{f}\ \ (\mathrm{pair}\ \ \mathrm{zero}\ \ \mathrm{one}))) \\
&= \ \mathrm{first}\ \ (\mathrm{f}\ \ (\mathrm{pair}\ \ \mathrm{one}\ \ \mathrm{two})) \\
&= \ \mathrm{first}\ \ (\mathrm{pair}\ \ \mathrm{two}\ \ \mathrm{three}) \\
&= \ \mathrm{two}
\end{aligned}
$$

How to define multiplication?

Recall that we have $n + m$, we want to define $n \times m$ recursively as

$$(\times) = \lambda nm.\,\textsf{if}(n = 0)\textsf{then}\ 0\ \textsf{else}\ (m + (n - 1) \times m)$$

we replaced **ite**, **iszero**, **add**, **pred** with syntactic sugar

Recursive function

Note that $(\times)$ is a free variable on the right-hand side. So we rewrite the term

$$(\times) = (\lambda f n m.\, \textsf{if}(n = 0)\textsf{then } 0 \textsf{ else } (m + f\ (n-1)\ m))(\times)$$

The right-hand side still contains the free variable $\textsf{fact}$. But we get a closed term, which we will abbreviate as F,

$$F = \lambda f n m.\, \textsf{if}(n = 0)\textsf{then } 0 \textsf{ else } (m + f\ (n-1)\ m)$$

now we have

$$(\times) = F\ (\times)$$

Now what can we do with F?

Iteration method

- First attempt: apply F with $\perp$

$$F(\perp) = \lambda nm.\,\text{if}(n = 0)\text{then } 0 \text{ else } (m + \perp)$$

Far away from what we want. But at least it is correct when $n = 0$

- Next attempt: apply F with $F(\perp)$

$$F(F(\perp)) = \lambda nm.\,\text{if}(n = 0)\text{then } 0 \text{ else } (m + F(\perp)\ (n-1)\ m)$$

$F(\perp)$ is correct when $n = 0$, so $F(F(\perp))$ is correct when $n = 0$ or $n = 1$

- Actually, $F^i(\perp)$ correctly calculates $n \times m$ for $n < i$
- how can we iterate this process?

Infinite reduction

Consider the following term

$$\omega = \lambda x.\, xx$$

Then we define the Ω combinator

$$\Omega = \omega\omega$$

which reduces to itself

$$(\lambda x.\, xx)(\lambda x.\, xx) \to_\beta (\lambda x.\, xx)(\lambda x.\, xx) \to_\beta \cdots$$

- Turing machine can also loop forever

Y combinator

The Y combinator is defined as

$$Y = \lambda f.\,(\lambda x.\,f(xx))(\lambda x.\,f(xx))$$

Repeatedly applying this equality gives:

$$Y\ F =_\beta F\ (Y\ F) =_\beta F\ (F\ (Y\ F)) =_\beta F\ (\cdots F\ (Y\ F) \cdots)$$

$Y\ F$ is also known as a fixed-point of F

Y combinator

Then we can define the multiplication function to be

$$(\times) = Y\ F$$

where

$$F = \lambda fnm.\,\textsf{if}(n = 0)\textsf{then}\ 0\ \textsf{else}\ (m + f\ (n - 1)\ m)$$

This is correct, because Y combinator iterates $F(F(\cdots))$ infinitely many times

Actually, applying the property $Y\ F = F\ (Y\ F)$

$$(\times) = \lambda nm.\,\textsf{if}(n = 0)\textsf{then}\ 0\ \textsf{else}\ (m + (n - 1) \times m)$$

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Fixed-point

In match, x is a fixed-point of a function f if

$$x = f(x)$$

for example, 2 is a fixed point of $f(x) = x^2 - 3x + 4$

Similarly, the fixed-point of F is the lambda term X such that

$$X = F\ X$$

Memoization

- y-combinator
- Consider the code for calculating fibonacci numbers

```
let rec fib = n => {
  switch n {
  | 0 | 1 => 1
  | _ => fib(n-1) + fib(n-2)
  }
}
```

- Tying the knot

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Memoization

```
let memofib = {
  let cache = Hashtbl.create(100)
  (n) => {
    switch Hashtbl.find_opt(cache, n) {
    | Some(x) => x
    | None => {
      let x = fib(n)
      Hashtbl.replace(cache, n, x)
      x
      }
    }
  }
}
```

Untying the knot

```
let myfib = (myfib,n)=>{
  switch n {
  | 0 | 1 => 1
  | _ => myfib(n-1)+myfib(n-2)
  }
}
```

- not recursive
- open recursion

Memoization

```
let memo  = anyFunc => {
  let cache = Hashtbl.create(100)
  let rec fix = (n) => {
    switch Hashtbl.find_opt(cache, n) {
    | Some(x) => x
    | None => {
      let x = anyFunc(fix,n)
      Hashtbl.replace(cache, n, x)
      x
      }
    }
  }
  fix
}
let memofib = memo(myfib)
```

Homework

- Implement the substitution function `N[v/x]`:
  `subst (N: lambda x: string, v: value) : lambda`
- Think about how substitution works on arbitrary terms, i.e. `N[M/x]` where `M` could contain free variables.
- Implement Church numberals and arithmetic functions using lambda calculus